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Many-to-One Stable Matching: Geometry and Fairness

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Baïou and Balinski characterized the stable admissions polytope using a system of linear inequalities. The structure of feasible solutions to this system of inequalities—*fractional* stable matchings—is the focus of this paper. The main result associates a geometric structure with each fractional stable matching. This insight appears to be interesting in its own right, and can be viewed as a generalization of the lattice structure (for integral stable matchings) to fractional stable matchings. In addition to obtaining simple proofs of many known results, the geometric structure is used to prove the following two results: First, it is shown that assigning each agent their "median" choice among all stable partners results in a stable matching, which can be viewed as a "fair" compromise; second, sufficient conditions are identified under which stable matchings exist in a problem with externalities, in particular, in the stable matching problem with couples.

1. Introduction. The stable marriage problem and its variants have been studied extensively over the last few decades. Beginning with the pioneering work of Gale and Shapley [17], this problem has captured the attention of researchers and practitioners in several disciplines such as computer science, economics, mathematics, and operations research. The multidisciplinary nature of these problems has led to a thorough understanding of many aspects, such as the design of efficient algorithms to find a stable matching, the structure of all solutions to a given stable matching instance, etc. In fact, by now there are several well-developed approaches to stable matching problems: the combinatorial/algorithmic approach as summarized in the books of Knuth and Goldstein [24] and Gusfield and Irving [18]; the linear programming approach initiated by Vande Vate [39], and further developed by Rothblum [36], Roth et al. [34], Teo and Sethuraman [38], Baïou and Balinski [6], and Fleiner [16]; the fixed-point approach of Subramanian [37] and Feder [12, 13]; an alternative fixed-point approach of Adachi [2]; yet another fixed-point approach due to Fleiner [15]; and a graph-theoretic approach due to Balinski and Ratier [7, 8] and Baïou and Balinski [5].

In addition to its elegant theory, a particularly appealing feature of the stable matching model is its applicability. In fact, starting with the work of Roth [28], applications to the National Resident Matching Program (NRMP) and related labor markets have given rise to interesting questions, resulting in a better understanding of the theory. Moreover, insights from this theory have been useful in the redesign of the stable matching algorithm used by the NRMP (Roth and Peranson [31]). An introduction to the theory of stable matchings with particular emphasis on applications to labor markets is elegantly summarized in Roth and Sotomayor [32]; recent work in this direction includes Roth and Peranson [30, 31], Cantala [11], and Klaus and Klijn [22].

This close interaction between applications and theory continues to this day and is an inspiration for the questions we study, described next.

1.1. Problem description, motivation, and results. It has long been recognized that the (natural) stable matching mechanisms suggested by the "proposal" algorithm of Gale and Shapley [17] are biased, i.e., they compute the *best* stable matching for *one* side of the market, which incidentally is also the worst stable matching for the other side of the market. From its inception until 1997, the NRMP used the hospital-optimal matching mechanism as the basis of its allocation. However, increasing pressure from student bodies and others[1] led to the redesign of the matching mechanism in 1998 (Roth and Peranson [30, 31]).

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[1] This took its clearest form in an exchange in the June 1995 issue of *Academic Medicine*, followed by position papers put out by the American Medical Students Association and others. Much of that discussion was oriented around results concerning "two-sided matching markets," of which the NRMP is a special case.

The inequitable treatment of the participants on different sides of the market persists in most known matching mechanisms. This has motivated the need for the design of fair stable matching mechanisms, which do not overtly favor one side of the market over the other. Considerable efforts have been devoted to finding "fair" stable matchings, including the *egalitarian* solution (minimize the sum of the ranks of the participants) and the *minimum-regret* solution (maximize the welfare of the participant worst-off in the matching) (Gusfield and Irving [18]). Nevertheless, these approaches are not satisfactory because they focus on socially optimal solutions and ignore the issue of fairness at the individual level.

Motivated by "procedural" fairness considerations, Klaus and Klijn [22] analyze three probabilistic stable matching mechanisms: *employment by lotto*, proposed by Aldershof et al. [4]; the *random order mechanism*, proposed by Roth and Vande Vate [35] and Ma [25]; and the *equitable random order mechanism*, proposed by Romero-Medina [27]. Their analysis shows that the three mechanisms may give completely different outcomes. They also note that the associated probability distribution for each of these mechanisms need not be uniform on the set of stable matchings; in fact, not all stable matchings can arise from these random matching mechanisms. Klaus and Klijn [22] construct an example and showed that a stable matching that constitutes a *perfect compromise* between contrary preferences on both sides of the market may never result from random matching mechanisms. Curiously, the perfect compromise solution identified there is the *median* solution (where the agents are assigned to their median stable partners), whose existence had in fact been established earlier in Teo and Sethuraman [38] for the one-to-one stable matching problem. The existence of such a solution is not obvious: Consider the assignment obtained by pairing each agent with his or her median stable partner; that such an assignment should be a matching is itself surprising, so it is quite amazing that the resulting solution is not only a matching but is also stable! In this paper, we establish the existence of such a median solution concept for the many-to-one model, thus proposing a new approach to address the asymmetry often observed in the stable matching problem. This result, as well as the earlier result in Teo and Sethuraman [38], can be viewed as a generalization of an observation first made by Conway (Gusfield and Irving [18]) on the lattice structure of stable matchings.[2]

Several generalizations and extensions of the stable matching problem have been studied in the past because of potential applications to NRMP and other labor markets. Of particular interest is the one in which certain pairs of students ("couples") would like to be assigned to universities that are geographically close together. We identify natural conditions on the preferences under which the couples problem can be handled effectively. Note that, in this case, a stable matching solution may not even exist. Furthermore, Klaus et al. [23] show that in the presence of couples, the current NRMP algorithm may fail to converge to a stable matching!

The common theme underlying our results is a refined understanding of the structure of "fractional" stable matchings. Baïou and Balinski [6] characterize the convex hull of all stable admissions solutions using linear inequalities in the natural assignment variables. We use the Baïou-Balinski formulation to show that the fractional stable admission solutions can be decomposed into convex combination of (integral) stable admission solutions in a simple way. As a by-product, our approach gives a simple visual proof of the integrality of the Baïou-Balinski formulation.

The main results in this paper are derived using a "bin-packing" theorem, obtained by packing a given fractional solution in a particular way. Interestingly, this approach can be viewed as a continuous analog of the bin-packing theorem used by Fleiner [15] to describe the polytope of the more general many-to-many stable matchings. However, in the approach used by Fleiner, the underlying inequalities are obtained through an iterative algorithm; the description of the polytope is thus *implicit*. The Baïou-Balinski formulation is explicit and is thus a more convenient and natural approach for our goal.

1.2. Organization of the paper. We introduce the stable admissions polytope in §2. In §3, we provide a geometric structure to the fractional stable admissions solutions. Section 4 uses this structure to design a fair stable matching mechanism for the many-to-one stable matching problem. We also describe how the stable matching problem with couples can be addressed using this approach. We end with a brief summary.

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[2] Since the first version of this manuscript, Fleiner [14] and Klaus and Klijn [21] (independently of each other) have provided (similar) proofs of the existence of the median stable matching, using the lattice structure. We deduce the existence of a median solution by working with fractional solutions, whereas they work directly with the set of (integral) many-to-one stable matchings. Their proof can be viewed as a "discrete" analog of ours. An anonymous referee observed the following simple proof of existence: construct the natural expanded one-to-one instance obtained by making q_u copies of university u, each copy of u having the same preferences as u itself. To break ties, we allow the students to rank the copies in an arbitrary but fixed manner. The stable matchings for the expanded one-to-one instance can be shown to be in one-to-one correspondence with the stable matchings for the many-to-one problem. This allows us to use the results derived in Teo and Sethuraman [38], on top of Theorem 4.1, to derive a simple proof for the existence of the median stable matching solution.

2. The stable admissions model. An instance of the stable admissions problem consists of two sets of agents, the set $U = \{u_1, u_2, \ldots, u_{|U|}\}$ of "universities" and the set $A = \{a_1, a_2, \ldots, a_{|A|}\}$ of "students." Each agent has a strict, transitive, preference ordering of the *acceptable* agents on the other side of the market, i.e., those agents on the other side of the market that it prefers to remaining unmatched. Let $\Gamma \subseteq U \times A$ denote the set of acceptable pairs. We assume without loss of generality that (i) u finds a acceptable if and only if a finds u acceptable, and in this case, we say that (u, a) is an *acceptable* pair; and (ii) university u finds at least q_u students acceptable, and each student finds at least one university acceptable. Associated with university u is a positive integer q_u representing its quota, the interpretation being university u is allowed to admit up to q_u students. Note that we consider the somewhat restrictive model of *responsive* preferences in which the universities have preferences over *individual* students, not over *groups* of students. We refer the reader to Roth and Sotomayor [32, Chapter 6] for interesting discussions on this issue.[3]

A matching μ for a stable admissions problem is a subset of Γ such that each university u appears in at most q_u pairs and each student a appears at most once in μ. For convenience, we let $\mu(a)$ represent the university that student a is assigned to, and $\mu(u)$ represent the *set* of students assigned to university u. We let $\mu(a) = \{a\}$ if student a is unmatched in μ.

A matching μ is *stable* if there is no incentive for any pair (u, a) to deviate from μ. That is, there is no pair $(u, a) \in \Gamma$ such that student (i) student a prefers university u to $\mu(a)$ and (ii) university u prefers to add student a to its set of students, possibly at the expense of another (less-preferred) student. (If a is unmatched in μ, then (i) is trivially satisfied.)

By the classical result of Gale and Shapley [17], every instance of the stable admissions problem admits a stable matching. The stable admissions problem with $q_u = 1$ for all $u \in U$ is called the *stable marriage* problem.

2.1. Definitions and notation. The symbols $>_u$ and $>_a$ will indicate the preference orderings of university u and student a, respectively; in particular, $a >_u b$ iff university u prefers student a to student b, and $a \geq_u b$ iff $a >_u b$ or $a = b$. The *shaft*, $S(u, a)$, is defined for every $(u, a) \in \Gamma$ as

$$S(u, a) = \{(u, b) \in \Gamma : b \geq_u a\}.$$

The *tooth*, $T(u, a)$, is defined for every $(u, a) \in \Gamma$ as

$$T(u, a) = \{(v, a) \in \Gamma : v \geq_a u\}.$$

For any set of q_u students $\{a, a_1, \ldots, a_{q_u-1}\}$ with $a_i >_u a$ for all i, the *comb* $C(u, a, a_1, \ldots, a_{q_u-1})$ is the union of the shaft $S(u, a)$, the tooth $T(u, a)$, and the teeth $T(u, a_i)$, $i = 1, \ldots, q_u - 1$. Such a comb is said to be based at (u, a). Let C_u be the collection of all combs based at (u, a), for some $a \in A$, and let $\mathscr{C} = \bigcup_{u \in U} C_u$. To any stable admissions solution μ, we can associate a $|\Gamma|$-vector x_μ (or x, if the stable matching μ is clear from the context) by letting $x_\mu(u, a) = 1$ if $(u, a) \in \mu$, and $x_\mu(u, a) = 0$ otherwise. For any $\hat{\Gamma} \subseteq \Gamma$, we let $x(\hat{\Gamma}) = \sum_{(u,a) \in \hat{\Gamma}} x(u, a)$. A pair $(u, a) \in \Gamma$ blocks μ if (i) u prefers a to at least one of its assigned students in μ, or if u is assigned fewer than q_u students; and (ii) a prefers u to its assigned university in μ, or if a is unmatched. We can express this condition using the notions of shaft and tooth defined earlier:

$$(u, a) \quad \text{blocks} \quad \mu \quad \text{iff} \quad x_\mu(S(u, a)) \leq q_u - 1 \quad \text{and} \quad x_\mu(T(u, a)) = 0. \tag{1}$$

The convex hull of the incidence vectors of all stable admissions solutions is the stable admissions polytope (of that instance), discussed next.

2.2. The stable admissions polytope. Let $C(u, a, a_1, \ldots, a_{q_u-1})$ be a comb in C_u. Every stable admissions solution x must satisfy the *comb inequality*

$$x(C(u, a, a_1, \ldots, a_{q_u-1})) \equiv x(S(u, a)) + x(T(u, a) \setminus \{(u, a)\}) + \sum_{i=1}^{q_u-1} x(T(u, a_i) \setminus \{(u, a_i)\}) \geq q_u, \tag{2}$$

first described by Baïou and Balinski [6]. Otherwise, $x(S(u, a)) < q_u$, and $x(T(u, a')) = 0$ for some $a' \in \{a, a_1, a_2, \ldots, a_{q_u-1}\}$. As $a' \geq_u a$, $x(S(u, a')) \leq x(S(u, a)) \leq q_u - 1$. By (1), the pair (u, a') is a blocking pair

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[3] It is conceivable that the results in this paper remain valid for more general and realistic classes of preferences over groups of students (cf. Abdulkadiroğlu [1]). The proof techniques needed for the extension, however, will be entirely different because our argument relies on a characterization of the convex hull of the set of stable solutions.

in x. It is a simple matter to verify that for any unstable matching μ, one can find a violated comb inequality: Let (u, a) be a blocking pair, and $\hat{a}$ be the student who is ranked q_uth in u's preference list. If u prefers $\hat{a}$ to a, any comb based on (u, a) that includes the teeth $T(u, a')$ for all $(u, a') \in \mu$, $a' >_u a$ is violated; otherwise, the (only) comb based on $(u, \hat{a})$ is violated. These considerations show that the integer programming problem (see Baïou and Balinski [6])

$$\sum_{u:\,(u,a)\in\Gamma} x_{u,a} \leq 1, \quad \forall a \in A, \tag{3}$$

$$\sum_{a:\,(u,a)\in\Gamma} x_{u,a} \leq q_u, \quad \forall u \in U, \tag{4}$$

$$x(C) = \sum_{(u,a)\in C} x_{u,a} \geq q_u, \quad \forall C \in C_u, \ u \in U, \tag{5}$$

$$x_{u,a} \in \{0, 1\}, \quad \forall (u, a) \in \Gamma \tag{6}$$

is an exact formulation of the stable admissions problem. Let P_{FSA} denote the set of all solutions to the associated (fractional) linear programming relaxation in which the constraints $x_{u,a} \in \{0, 1\}$ are replaced by $x_{u,a} \geq 0$. Let P_{SA} be the convex hull of all stable matchings to the stable admissions problem. Clearly, $P_{SA} \subseteq P_{FSA}$. The main result of Baïou and Balinski [6] is that, in fact, $P_{SA} = P_{FSA}$.

In the next section we study the structure of fractional solutions of the stable admissions polytope; we exploit this structure in the following sections to derive a host of interesting results, including a simple proof that $P_{SA} = P_{FSA}$.

3. Geometry of fractional stable admissions.

3.1. Decomposition. To motivate our main result, it is useful to think of each university u as owning q_u "bins," each of size (=height) 1; each bin represents a seat and is indexed by the pair (i, u). Each $x_{ua} > 0$ will be treated as an "item" to be packed into one of the bins owned by university u. Bins are filled in the usual way (bottom to top); all items are packed in decreasing preference order of the university, following the procedure described next.

In phase 0, each university u packs its (at most) q_u best items, (at most) one in each of its bins. (If u is fractionally matched with fewer than q_u students, then some of its bins will remain unfilled.) Bins that are completely empty at the end of this phase are excluded from the rest of the packing process. Phase t (for $t = 1, 2, \dots$) of the procedure consists of (a) identifying the set, L_t, of bins with the maximum available space; and (b) assigning one item to each of the bins in L_t. The assignment of the items to the bins within a phase proceeds in a sequence of *steps*, indexed by $l = 1, 2, \dots, |L_t|$. By our assumption, if bin $(i, u) \in L_t$ is considered in step l, the university u's best remaining item is packed into it. We have not specified the order in which the bins are examined within each phase. As it turns out, the order does not matter.

At the end of any *step* of the packing procedure, let $M_{i,u}$ be the set of students assigned to bin i of university u; and let $a_{i,u}$ be university u's least-preferred student in $M_{i,u}$. In terms of our bin-packing analogy, $a_{i,u}$ is the student occupying the topmost position in bin (i, u). Our main result is the following.

THEOREM 3.1. *Consider any $x \in P_{FSA}$. Let $M_{i,u}$ be the set of students assigned to bin i of university u at the end of any step of the packing procedure. Let $a_{i,u}$ be university u's least-preferred student in $M_{i,u}$. Then,*
(a) *For all (i, u) with $M_{i,u} \neq \varnothing$,*

$$x(M_{i,u} \setminus a_{i,u}) + x(T(u, a_{i,u})) = x(M_{i,u}) + x(T(u, a_{i,u}) \setminus a_{i,u}) = 1;$$

(b) *If item (v, a) is packed but (u, a) is not, then a prefers u to v.*
(c) *At the end of any phase, the $a_{i,u}$ are all distinct. In particular, for each $a \in A$ with $\sum_{v \in U} x_{va} > 0$, there is some (i, u) such that $a \equiv a_{i,u}$.*

Theorem 3.1(a) plays a similar role to the stability constraints in the one-to-one stable marriage problem. Interestingly, the packing procedure decomposes the q_u positions for university u implicitly into q_u copies, each of quota 1, and identifies the set of students used to fill the ith position in university u. This essentially reduces the stable admission problem to a one-to-one stable marriagelike problem. Note that the universities pack the students into the bins in *decreasing* preference order, but Theorem 3.1(b) states that the packing process assigns universities to students in *increasing* preference order. Theorem 3.1(c) states that at the end of each phase, the items at the top of each bin forms a valid assignment.

Before proving the theorem, it will be useful to consider an illustrative example.

3.2. Example. Consider the following example, adapted from §1.6.5 of Gusfield and Irving [18]. There are 5 universities and 11 students, with the preference lists and the quotas given in Tables 1 and 2. There are a total of seven stable matchings for this particular problem instance, and these are listed in Table 3.

Let x_{M_i} be the (integral) stable matching corresponding to M_i in the stable admission polytope, and let $x = (\sum_{i=1}^{7} x_{M_i})/7$ be a fractional solution in the stable matching polytope. We illustrate the packing procedure for this solution x.

Phase 0. Initially, all the bins are empty, so L_1 consists of all the bins; each of these bins will have an item in it at the end of the first phase. Recall that items are packed by the universities in decreasing preference order.

Consider university u_1, which owns four bins. Its four best students in x are a_9, a_{11}, a_5, and a_4, with the corresponding "values" being $3/7$, $1/7$, 1, and $2/7$, respectively. Thus, the four bins owned by u_1 contain these four items as shown in Figure 1. For convenience, we may also assume that each student also owns a bin in which items are packed: whenever item (u, a) is packed into bin (i, u), we may also assume that item (u, a) is packed into the bin owned by student a. Thus, students a_9, a_{11}, a_5, and a_4 each pack an item into their bins. In terms of this picture, it is easy to interpret the three statements of Theorem 3.1: part (b) simply says that this packing, when viewed from the point of view of the students, proceeds in increasing preference order; given this, part (a) says that the *occupied* space in any bin with student a on top plus the *available* space in student a's bin is exactly 1; and part (c) says that at the end of any phase, the students on top of the bins owned by the universities are all distinct. All of these properties are easily verified from Figure 1, which shows the state of the bins at the end of Phase 0.

Phase 1. The set L_2 will consist of all the bins that have the maximum available space; from Figure 1, it is clear that L_2 consists of Bin 2 of u_1, Bin 1 of u_4, and the bin owned by u_5. During Phase 1, each of these three bins will have an item assigned to it. In fact, the top students in these bins will rearrange themselves, so that part (c) continues to hold. University u_1 will pack its most-preferred remaining items, which happens to be student a_{10} (value $6/7$); similarly, universities u_4 and u_5 will pack, respectively, items involving students a_{11} (value $4/7$) and a_1 (value $2/7$), respectively. The state of the bins at the end of Phase 1 is shown in Figure 2; it is easy to verify that parts (a)–(c) of Theorem 3.1 continue to hold.

End of packing process. By repeating the above packing procedure, the final state of the bins (after Phase 4) is shown in Figure 3.

3.3. Proof of Theorem 3.1. Let $x \in P_{FSA}$. We say that a particular university-student pair (u, a) is *matched* (under x) if $x_{ua} > 0$; we say that they are *fractionally matched*, if $0 < x_{ua} < 1$.

It will be useful to understand the conditions under which a particular university-student pair is fractionally matched.

LEMMA 3.1. *Suppose (k, l) is fractionally matched under x, i.e., $x_{kl} \in (0, 1)$. Then,*

$$\sum_{j \in A} x_{kj} = q_k, \quad and \quad \sum_{i \in U} x_{il} = 1.$$

PROOF. The necessary ingredients to prove this already appear in Baïou and Balinski [6]. Let $B_u(x)$ be u's q_u most-preferred students among the students it is matched to under x; if u is matched with fewer than q_u students, we let $B_u(x)$ be the set of all students that u is matched to. Let $A_u(x)$ be the set of all students for whom u is the most-preferred university they are matched to under x.

Two simple facts will be useful in the proof:

$$x(T(u, a)) = 1, \quad \forall a \in B_u(x), \tag{7}$$

TABLE 1. Preference lists for the students.

a_1	u_3	u_1	u_5	u_4	
a_2	u_1	u_3	u_4	u_2	u_5
a_3	u_4	u_5	u_3	u_1	u_2
a_4	u_3	u_4	u_1	u_5	
a_5	u_1	u_4	u_2		
a_6	u_4	u_3	u_2	u_1	u_5
a_7	u_2	u_5	u_1	u_3	
a_8	u_1	u_3	u_2	u_5	u_4
a_9	u_4	u_1	u_5		
a_{10}	u_3	u_1	u_5	u_2	u_4
a_{11}	u_5	u_4	u_1	u_3	u_2

TABLE 2. Quotas and preference lists for the universities.

Capacity											
(4)	u_1	a_3	a_7	a_9	a_{11}	a_5	a_4	a_{10}	a_8	a_6	a_1	a_2
(1)	u_2	a_5	a_7	a_{10}	a_6	a_8	a_2	a_3	a_{11}			
(3)	u_3	a_{11}	a_6	a_8	a_3	a_2	a_4	a_7	a_1	a_{10}		
(2)	u_4	a_{10}	a_1	a_2	a_{11}	a_4	a_9	a_5	a_3	a_6	a_8	
(1)	u_5	a_2	a_4	a_{10}	a_7	a_6	a_1	a_8	a_3	a_{11}	a_9	

TABLE 3. List of all stable matchings.

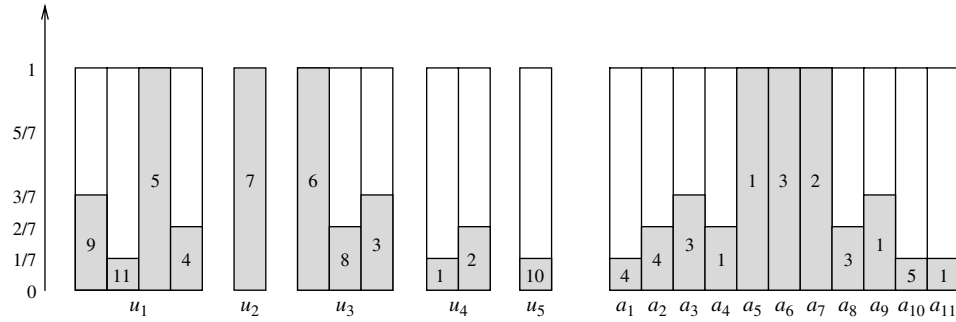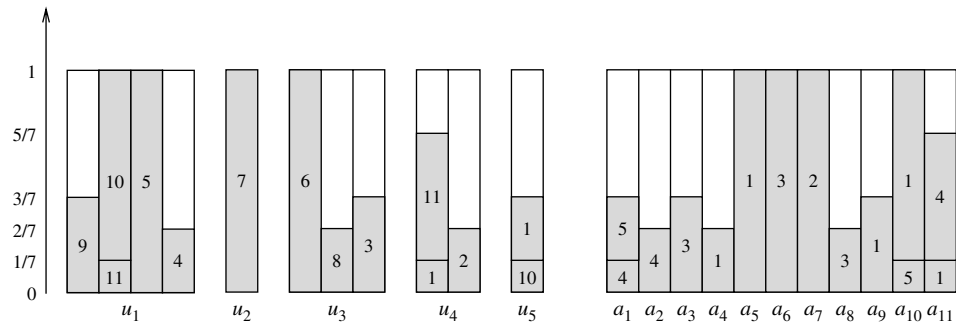| Matching	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
M_1	u_3	u_1	u_4	u_3	u_1	u_3	u_2	u_1	u_4	u_1	u_5
M_2	u_1	u_3	u_4	u_3	u_1	u_3	u_2	u_1	u_4	u_1	u_5
M_3	u_3	u_1	u_5	u_3	u_1	u_3	u_2	u_1	u_4	u_1	u_4
M_4	u_1	u_3	u_5	u_3	u_1	u_3	u_2	u_1	u_4	u_1	u_4
M_5	u_5	u_3	u_3	u_4	u_1	u_3	u_2	u_1	u_1	u_1	u_4
M_6	u_5	u_4	u_3	u_1	u_1	u_3	u_2	u_3	u_1	u_1	u_4
M_7	u_4	u_4	u_3	u_1	u_1	u_3	u_2	u_3	u_1	u_5	u_1

FIGURE 1. Bins at the end of Phase 0.
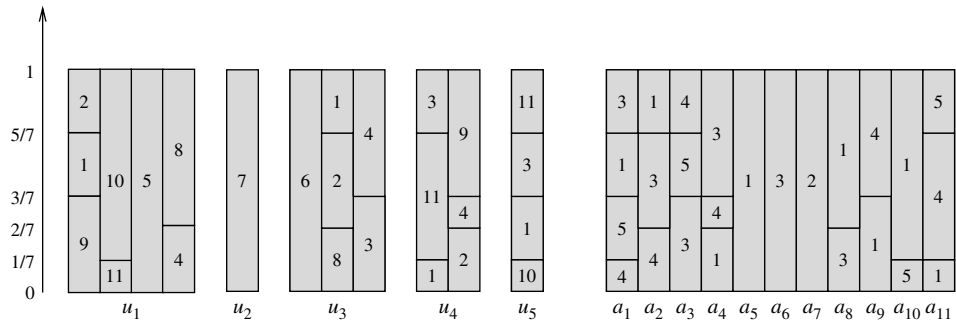

FIGURE 2. Bins at the end of Phase 1.

FIGURE 3. Bins at the end the packing procedure.

and

$$x(T(u, a)) = x_{ua}, \quad \forall a \in A_u(x). \tag{8}$$

The latter is immediate by definition. To see the former, consider the comb C involving u and the set of students $B_u(x)$, supplemented, if necessary, with additional students not matched to u to get q_u "teeth." Let $a_1, a_2, \ldots, a_{q_u}$ be these q_u students. By definition,

$$x(C) = \sum_{i=1}^{q_u} x(T(u, a_i)). \tag{9}$$

The LHS of Equation (9) is at least q_u by the comb inequality; whereas the RHS of Equation (9) is at most q_u by aggregating the appropriate student assignment constraints. Therefore,

$$x(T(u, a_i)) = 1. \tag{10}$$

In particular, $x(T(u, a)) = 1$ for each $a \in B_u(x)$.

We prove next that $|A_u(x)| = |B_u(x)|$ for all $u \in U$: Clearly, $B_u(x) \cap B_v(x) = \varnothing$ for $u \neq v$, otherwise there will be some $a \in B_u(x) \cap B_v(x)$ with $x(T(u, a)) = 1$ and $x(T(v, a)) = 1$. Hence,

$$\sum_{u \in U} |A_u(x)| \geq \sum_{u \in U} |B_u(x)|.$$

Therefore, to prove $|A_u(x)| = |B_u(x)|$, it is enough to show that $|A_u(x)| \leq |B_u(x)|$ for all $u \in U$.

If $|B_u(x)| < q_u$, then $B_u(x)$ contains all the students who are matched to u. It is thus clear that $|A_u(x)| \leq |B_u(x)|$. Let $|B_u(x)| = q_u$, and suppose $|A_u(x)| > |B_u(x)|$. Consider the comb C with the teeth chosen from u's q_u best students from $A_u(x)$. By Equation (8), all the positive x components of comb C involve university u; moreover, because some member of $A_u(x)$ is excluded from comb C, $x(C) < q_u$ by the university assignment constraint, which violates the comb inequality for comb C. Therefore, $|A_u(x)| \leq |B_u(x)|$.

Based on the preceding discussion, $|A_u(x)| = |B_u(x)|$ for each $u \in U$. The lemma now follows from the following reasoning:

(a) If $|B_u(x)| < q_u$, then $|A_u(x)| = |B_u(x)|$ and Equations (7) and (8) imply $x_{ua} = 1$ for all $a \in B_u(x)$. Therefore, any u with $|B_u(x)| < q_u$ cannot have any fractionally matched students under x.

(b) Any fractionally matched student a belongs to some $A_u(x)$. Because the $A_u(x)$ are all disjoint and the $B_v(x)$ are all disjoint, and since $\sum_u |A_u(x)| = \sum_v |B_v(x)|$, a must also appear in $B_v(x)$ for some $v \in V$. By Equation (7), $x(T(v, a)) = 1$; in particular, $\sum_{i \in U} x_{ia} = 1$.

(c) If $x_{ua} \in (0, 1)$, then $|B_u(x)| = |A_u(x)| = q_u$. The comb inequality constraint on the comb $C(u, A_u(x))$ and the university assignment constraint for u show that u must meet its quota exactly. $\square$

Lemma 3.1 ensures that any university u with unfilled bins is only integrally matched to its students. In our packing procedure, such universities and the students to whom they are matched pack their items only at Phase 0, after which they play no role in the packing process. The residual instance now consists of all the universities that fill their quota, and the students with whom they are matched.

In the rest of this section, we can thus assume that every agent is "completely" matched; that is, $\sum_{u \in U} x_{ua} = 1$ for each $a \in A$, and $\sum_{a \in A} x_{ua} = q_u$ for each $u \in U$.

PROOF OF THEOREM 3.1. Our proof proceeds by induction on t. First, consider the case $t = 0$. Because we start with a set of empty bins, all the bins will be in L_0; at the end of the phase, it is clear that university u will have (fractionally) packed its q_u most-preferred items, one in each of its bins. Let $B_u(x)$ be university u's q_u most-preferred students it is matched to under x.

Part (a) of the theorem now follows from observing that $M_{i,u} = a_{i,u}$. Part (b) is true because $x(T(u, a_{i,u})) = 1$, and $x_{u, a_{i,u}} > 0$ for all (i, u), and preferences are strict. Moreover, if $a_{i,u} = a_{j,v} = a$, then either $x(T(u, a)) < 1$ or $x(T(v, a)) < 1$, depending on whether $v >_a u$ or $u >_a v$; so the $a_{i,u}$ are all distinct at the end of Phase 0, establishing the first statement of part (c). Because the number of students is the same as the total number of bins, the second statement of part (c) follows.

Suppose the theorem is true up to the end of phase t. We now establish the theorem for phase $t + 1$. Let α be the maximum space available in any bin during phase $t + 1$. Consider any item (u, a) that has not yet been packed. From part (c), $a \equiv a_{j,v}$ for some bin (j, v); from parts (a) and (b), we have

$$x(T(u, a)) \leq x(T(v, a)) - x_{v, a} = 1 - x(M_{j, v}) \leq \alpha, \tag{11}$$

where the last inequality follows because each bin has at most α available space. Moreover,

$$x(T(u, a)) = \alpha \quad \text{implies} \quad (j, v) \in L_{t+1}. \tag{12}$$

Suppose $(i, u) \in L_{t+1}$ is chosen in any step of phase $t+1$, and item x_{ua} is to be packed into bin (i, u) next. Consider the comb

$$C_{u,a} \equiv C(u, a, a_{1,u}, a_{2,u}, \ldots, a_{i-1,u}, a_{i+1,u}, \ldots, a_{q_u,u}),$$

which contains

- the teeth $T(u, a)$, $T(u, a_{j,u})$ for $j = 1, 2, \ldots, i-1, i+1, \ldots q_u$; and
- the shaft $S(u, a) = \{(u, a)\} \cup M_{1,u} \cup M_{2,u} \cup \cdots \cup M_{q_u,u}$.

By the comb inequality, we have

$$q_u \leq x(C_{u,a}) = \sum_{j: j \neq i} x(T(u, a_{j,u})) + x(T(u, a)) + x(M_{i,u}) + \sum_{j: j \neq i} x(M_{j,u} \setminus \{a_{j,u}\}).$$

By induction, for $j \neq i$,

$$x(T(u, a_{j,u})) + x(M_{j,u} \setminus \{a_{j,u}\}) = 1,$$

which reduces the comb inequality to

$$x(T(u, a)) + x(M_{i,u}) \geq 1. \tag{13}$$

Because $(i, u) \in L_{t+1}$ and has not been packed earlier, $x(M_{i,u}) = 1 - \alpha$; from Equation (11), $x(T(u, a)) \leq \alpha$. Therefore, the (reduced) comb inequality (13) must be satisfied as an equality. We next pack (u, a) in bin (i, u), and update $M_{i,u}$ and $a_{i,u}$, and observe that the reduced comb equality can be written as

$$x(T(u, a_{i,u})) + x(M_{i,u} \setminus a_{i,u}) = 1,$$

establishing part (a). Part (b) follows because if item (u, a) is not packed but item (v, a) is packed, then from Equation (11),

$$x(T(u, a)) \leq \alpha \leq x(T(v, a)).$$

Hence, a prefers u to v.

Observe that $x(T(u, a)) = \alpha$ for every item (u, a) packed in phase $t+1$, and these are the least-preferred items in their respective bins; this implies the least-preferred items in the newly packed bins are all distinct. From (12), if a bin (i, u) was not selected in phase $t+1$, its least-preferred student could not have been part of any item packed in phase $t+1$. Therefore, the $\{a_{i,u}\}$ remain distinct at the end of phase $t+1$. $\square$

3.4. Consequences. We begin this section by noting that the "rural hospitals" theorem, originally due to Roth [29], is an easy consequence of Lemma 3.1.

THEOREM 3.2. *The set of universities that fill their position is the same in all stable matchings, as is the set of students assigned to some university; moreover, any university that does not fill its quota is assigned the same set of students in every stable matching.*

PROOF. Let x be the average of all the stable matchings to the given stable admissions problem. Clearly, $x \in P_{FSA}$. Let u be a university that does not fill its quota in some stable matching. Then, $\sum_{a \in A} x_{ua} < q_u$. By case (c) in the proof of Lemma 3.1, $|B_u(x)| < q_u$, for otherwise, u must meet its quota exactly. Applying Lemma 3.1, part (a) to x, we see that $x_{ua} = 1$ for all $a \in B_u(x)$, so u must be matched to the same set of students in all stable matchings. Case (b) of Lemma 3.1 shows that the set of students assigned to a university is the same in all stable matchings. $\square$

Let $x \in P_{FSA}$. Because our goal is to highlight the structure of *fractional* solutions in P_{FSA}, we ignore universities that do not fill their quotas. As the proof of Lemma 3.1 and Theorem 3.2 show, such universities are assigned to the same set of students in every $x \in P_{FSA}$ and can thus be safely ignored.

For any $\alpha \in [0, 1)$, let

$$I_u^\alpha(x) = \{a: 1 - x(T(u, a)) \leq \alpha < 1 - x(T(u, a)) + x_{ua}\}, \tag{14}$$

and $I^\alpha(x) = \bigcup_{u \in U} I_u^\alpha(x)$. By part (a) of Theorem 3.1, $I_u^\alpha(x)$ consists of those students whose assignment to their "bin" in university u changes the occupied space in that bin from at most α to some quantity strictly greater than α.[4] Because each university fills all its bins, the following corollary of Theorem 3.1 is immediate.

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[4] Geometrically, if we lay out the items using our bin-packing approach, to obtain a figure resembling Figure 3, then $I^\alpha(x)$ consists of those items in the layout that intersect with a horizontal line at $y = \alpha$.

COROLLARY 3.1.   *For any $x \in P_{FSA}$ and any $\alpha \in [0, 1)$, $|I_u^\alpha(x)| = q_u$.*

REMARK.   Baïou and Balinski [6] observed that $I_u^0(x) = I_u^{1^-}(x) = q_u$ for any stable admissions solution x; Corollary 3.1 generalizes these observations.

Roth et al. [34] proved the following result for the stable *marriage* polytope using a primal-dual argument.

LEMMA 3.2.   *Let $q_u = 1$ for all $u \in U$. For any $x \in P_{FSA}$, $x_{u,a} > 0$ implies*

$$x_{u,a} + x(S(u,a)\backslash(u,a)) + x(T(u,a)\backslash(u,a)) = 1.$$

This result was crucial in proving that P_{FSA} is exactly the convex hull of all stable marriage solutions Roth et al. [34] and Teo and Sethuraman [38]. The following result generalizes Lemma 3.2 to the stable admissions setting; its proof is implicit in the proof of Theorem 3.1 and is therefore omitted.

LEMMA 3.3.   *For any $x \in P_{FSA}$, and any $\alpha \in [0, 1)$, let $C_{\alpha,u}$ denote the comb obtained by using the q_u students in $I_u^\alpha(x)$ as the bases for the teeth in the comb. Then, $x(C_{\alpha,u}) = q_u$.*

We next show how the main result of Baïou and Balinski [6] follows as an immediate consequence of Theorem 3.1.

THEOREM 3.3.   *The inequalities* (3)–(6) *determine the stable admissions polytope P_{SA}.*

PROOF.   Given any $x \in P_{FSA}$, generate a number $\alpha \in (0, 1]$ uniformly at random; for each university u, assign the q_u students in $I_u^\alpha(x)$ to university u to get an assignment μ_α. By part (c) of Theorem 3.1, no student is assigned to two different universities under μ_α, so that μ_α is a matching. Let $v = \mu_\alpha(b)$. We show next that μ_α is stable.

Consider any student b whom u strictly prefers to at least one of its assigned students (say a) under μ_α. It is possible that $x(u, b) = 0$. Consider the set $S^* = \{c: x(u, c) > 0, c <_u b\}$, and let b' be the most preferred student in S^*, according to u. Because $a \in S^*$, we have $a \leq_u b'$.

The item (u, b') would have been packed not later than item (u, a) by the packing procedure; moreover, at the *beginning* of the phase, say t, in which (u, b') was packed, the *space available* in the bin occupied by student b' at university u, say $1 - \beta$, must not be smaller than $1 - \alpha$, otherwise u cannot prefer b' to a.

If item (u, b') is packed in phase $t > 0$, then by Theorem 1, we can always pick a comb C based at (u, b') that is tight: $x(C) = q_u$. C can be obtained by choosing the least-preferred student in each bin, at the moment when item (u, b') is packed. We can form a new comb C' by shifting the base from (u, b') to (u, b). Note that $x(S(u, b)) = x(S(u, b')) - x(u, b')$. Using the comb inequality $x(C') \geq q_u$, and the fact that $x(C) = q_u$, we have

$$x(T(u, b)) \geq x(T(u, b')) = 1 - \beta \geq 1 - \alpha.$$

Hence, $v \geq_b u$.

If item (u, b') is packed in phase $t = 0$, then $b' \in B_u(x)$, i.e., (u, b') is one of the q_u most-preferred items according to u. Note that the comb C formed by the students in $B_u(x)$ is tight: $x(C) = q_u$. We can replace the most-preferred student in $B_u(x)$ by b to obtain a new comb C', with $x(C') \geq q_u$. Because $x(C')$ is simply the sum of the length of the q_u teeth, we have $x(T(u, b)) \geq 1$. Hence, again $v \geq_b u$.

The matching μ_α obtained is thus stable. By construction, the expected value of $\mu_\alpha(u, a)$ is $x_{u,a}$. Hence, any $x \in P_{FSA}$ can be written as $E(\mu_\alpha)$, i.e., a convex combination of stable matching solutions. Integrality of the polytope is now immediate (Teo and Sethuraman [38]).   □

4. Applications.   In the rest of this section, we use the geometry of fractional solutions of the stable admissions problem to address two natural applications: the design of a *fair* stable assignment mechanism and the problem of assigning *couples* to "universities."

4.1. Fairness.   It has long been recognized in the economics literature that there are subtle differences between "procedurally fair" and "outcome-fair" solution concepts (cf. Bolton et al. [9]). A procedurally fair mechanism is useful in instances when the fairness of the *process* is viewed as more important than the fairness of the *outcome*. People feel affirmed if the adopted procedures treat them with respect and dignity, making it easier to accept outcomes, even the ones they do not like. However, in a matching market like the NRMP, it is not clear whether a procedurally fair mechanism will be acceptable to all participants.

On the other hand, studies in which subjects divide resources between themselves and others have shown that most people preferred equality to either advantageous or disadvantageous inequality; see, for instance, Camerer

and Thaler [10]. An important debate in this literature concerns which outcome-fair solution concept is most appropriate for a particular situation, or whether any fair solution concept exists at all. The existence of a median stable matching in the one-to-one case suggests that outcome-fair resolution of the stable marriage problem may be possible. We will not attempt to formally justify the median stable marriage solution as an outcome-fair solution, but we note that in many experimental tests of bargaining solution concepts in the ultimatum game, Guth et al. [19] show that "equal splitting" is the dominant behavior observed amongst the participants. Interestingly, the median solution proposed in this section shares many characteristics with the equal-split option in the bargaining problem—with a proper geometric layout of all the stable marriage solutions along a line, the median solution is obtained by splitting at the middle, with universities (men) and students (women) at opposite ends. A formal description now follows.

Suppose there are exactly N stable matchings in an instance of the stable admissions problem, and let $x_1, \ldots, x_N$ be the stable solutions. For convenience, we assume N is odd. Is there a way to choose a solution that treats all the participants in a fair manner? The starting point for the results described in this subsection is the following result due to Roth and Sotomayor [33] (see also Gusfield and Irving [18, Theorem 1.6.4, p. 46] or Roth and Sotomayor [32, Theorem 5.27, p. 161]). We include here a simple proof based on Theorem 3.1.

THEOREM 4.1. *Given two stable matchings μ and μ', suppose university u is assigned nonidentical sets of students in μ and μ'. If u prefers its least-preferred student in $\mu \setminus \mu'$ to its least-preferred student in $\mu' \setminus \mu$, then u prefers all the students in μ to any student in $\mu' \setminus \mu$.*

PROOF. Let x_μ and $x_{\mu'}$ denote the (integral) stable admission solutions corresponding to the matchings μ and μ', respectively. Let $x = (x_\mu + x_{\mu'})/2$, and consider the packing procedure on the fractional solution x. Note that if student a is assigned to u in $\mu \cap \mu'$, then $x_{ua} = 1$. Hence, (u, a) must be among the q_u most-preferred items packed at the initial phase; otherwise, condition (a) of Theorem 3.1 will be violated.

Let s and s' be university u's least-preferred students in $\mu \setminus \mu'$ and $\mu' \setminus \mu$, respectively. By assumption, u prefers s to s'. Suppose u prefers a student in $\mu' \setminus \mu$ to some student in μ. Then, s cannot be one of u's q_u most-preferred students. From condition (c) of Theorem 3.1, the student s must be among the $q_{u'}$ most-preferred students of some university u' in the matching μ'. Because $x(T(u', s)) = 1$, s prefers u to u'. Now, because u prefers s to s', (u, s) forms a blocking pair for the stable assignment μ'. This is a contradiction. □

The key conclusion emerging from Theorem 4.1 is that any university either prefers all of its μ-partners to any of its μ'-partners who are not also μ-partners, or prefers all of its μ'-partners to any of its μ-partners who are not also μ'-partners. In the former case, we say that university u prefers μ to μ', and in the latter, that it prefers μ' to μ. In particular, the worst student for university u in its *different* stable assignments should all be different. Thus, each university can rank order all its stable assignments in decreasing order of its worst student.

Consider the set of student assignments university u receives under all stable solutions $x_1, \ldots, x_N$. By Theorem 4.1, u can rearrange these solutions such that its worst student in each of the solutions appears in decreasing preference order, that is, x_i appears before x_j if u prefers its worst student in x_i to its worst student in x_j. This induces a natural linear order on the set of solutions assigned to each university. We can thus define the *median choice* of each university: It is simply the median solution in the linear ordering of all stable solutions by the university.[5] Similarly, each student could rank order $x_1, x_2, \ldots, x_N$ in decreasing preference order based on the university to which she is assigned. In this case, her median choice will be the median university in this linear ordering. A natural question then is: If each university and student is given their median choice, do we get a stable matching? In fact, it is not even obvious that assigning the median choice to each agent gives a valid *assignment*: There is no reason why the same student should not be the median choice of two different universities, or why two students should not have the same university as their median choice. Remarkably, the median choices of each agent result in a *stable matching*!

In the rest of this section, we assume that the given instance satisfies the conditions that all universities filled their quota and all students are matched to a university. This is without loss of generality: Universities that do not fill their quotas have only one set of students in all stable matchings, so such universities and the students they are matched to are not particularly interesting in studying fairness issues in stable matchings. Let

$$x_f(u, a) = \frac{\sum_{j=1}^N x_j(u, a)}{N},$$

where $x_1, \ldots, x_N$ are the set of all stable matchings to the given instance of the stable admissions problem.

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[5] If N is even, the median choice is not unique but consists of two candidate solutions. Fortunately, the argument in this section can be easily modified to handle this case.

THEOREM 4.2. *Let $M_{i,u}^f$ be the set of students assigned to bin i of university u by the packing procedure applied to the fractional stable matching x_f. Then, each stable matching contains exactly one student from each $M_{i,u}^f$.*

PROOF. The result is obviously true for all universities that do not fill their quota: Such universities are matched to the same set of students in all stable matchings, and hence in x_f as well. Let u be a university that admits q_u students in some (and hence all) stable matching. Clearly, $\sum_{a: a \in A} x_f(u, a) = q_u$. Let $\mu_1 >_u \mu_2 >_u \cdots >_u \mu_K$ be the distinct sets of students that are assigned to university u in some stable matching. Note that $|\mu_i| = q_u$ for all i.

We examine how the packing procedure packs u's bins starting with the fractional solution x_f. As before, the proof is by induction on the number of phases; the only change is that we focus only on those phases that pack at least one of u's bins. (Our numbering of the phases also reflects this: Phase i is the ith time university u's bins are packed by the packing procedure.) By Theorem 4.1, all the items associated with μ_1 are packed in Phase 1, and each such item occupies a distinct bin; therefore, the result is true at the end of Phase 1. Which items are packed during Phase 2? Clearly, u's most-preferred items that have not yet been packed are those in $\mu_2 \setminus \mu_1$, and these will be packed in Phase 2. It is also clear that the bins into which these items are packed each contain one item of $\mu_1 \setminus \mu_2$: The remaining bins must contain items that are in both μ_1 and μ_2, so their values in x_f will be strictly larger than the values of the items in $\mu_1 \setminus \mu_2$. Thus, at the end of Phase 2, the items of $\mu_2 \setminus \mu_1$ will each occupy a bin that now contain the items in $\mu_1 \setminus \mu_2$; note that the stable assignments μ_1 and μ_2 contain exactly one item from each bin.

In Phase 3, the items in $S = \mu_3 \setminus (\mu_1 \cup \mu_2)$ will be packed. We claim that, in fact, $S = \mu_3 \setminus \mu_2$. Otherwise, there must be a student a such that $a \in \mu_1$, $a \in \mu_3$, but $a \notin \mu_2$. Because u prefers μ_1 to μ_2, u must prefer a to any student in $a' \in \mu_2 \setminus \mu_1$; and because u prefers μ_2 to μ_3, u must prefer a' to any student in $\mu_3 \setminus \mu_2$. In particular, u must prefer a' to a, which is a contradiction. The bins with maximum available space are clearly those containing the items in $T = \mu_2 \setminus \mu_3$. Because S and T have the same cardinality, and because each element of T already appears in a distinct bin, each item in S is assigned, by the packing procedure, to a bin containing an item of T. Now, each of μ_1, μ_2, and μ_3 contains exactly one item from each bin. (This is because only the items in S are newly assigned, none of which can be present in $\mu_1 \cup \mu_2$.)

We are now ready to do an inductive proof. At phase i, let $S = \mu_i \setminus \mu_{i-1}$, $T = \mu_{i-1} \setminus \mu_i$; clearly,

$$S = \mu_i \setminus \{\mu_1 \cup \mu_2 \cup \cdots \cup \mu_{i-1}\}.$$

Also, S and T are nonempty and have the same cardinality; all items in T are in distinct bins, so the items in S are assigned to distinct bins by the packing procedure, each containing a distinct item in T. Moreover, we have now ensured that the stable assignments $\mu_1, \mu_2, \ldots \mu_i$ all contain exactly one item from each bin. This is because only the items in S are newly assigned, none of which can be present in any of the assignments $\mu_1, \mu_2, \ldots, \mu_{i-1}$. $\square$

REMARKS. Theorem 4.2 has also been derived independently by Fleiner [16], using somewhat similar methods. In the proof of Theorem 3.1, we showed how all the fractional partners of any university u in a solution $x \in P_{FSA}$ can be partitioned into q_u sets. Theorem 4.2 shows something stronger: that one can in fact construct a *single* partition of the students that works for *all* (fractional) solutions. Moreover, we can construct an associated stable marriage instance whose stable solutions will be in a one-to-one correspondence with the set of all stable admissions solutions of the original instance. The advantage of such a construction is clear: All the results on the stable admissions problem can thus be reduced to that of the stable marriage problem by appealing to this transformation. Indeed, given this decomposition, it is a simple matter to find a compact linear formulation to express the convex hull of all stable matchings. The disadvantage is that constructing such a decomposition entails knowing the entire set of stable assignments for each university u.

The geometric structure of x_f is extremely useful, as we can now immediately read off many possible stable admission solutions from the layout.

- *University-optimal solution.* Match u to the q_u students in $I_{0^+}(u)$. The solution obtained is university optimal (best q_u possible students are assigned), but each student is assigned her worst possible stable university. In the example in Figure 3, this corresponds to choosing all items touching the bottom of every bin in the figure.

- *Student-optimal solution.* Match u to the q_u students in $I_1(u)$. The students are assigned to their best-possible stable university (i.e., student optimal), and the universities are assigned to their worst-possible set of students under any stable solution. In the example in Figure 3, this corresponds to choosing all items touching the top of every bin in the figure.

• *Median solution.* If N is odd, we can define the *median* stable admission solution as follows: Match u to the q_u students in $I_{0.5}(u)$. The students are thus matched to their median stable university (counting multiplicity). By the packing procedure of §3.1 and Theorem 4.2, it is easy to see that the assignment corresponding to $I_{0.5}(u)$ is exactly the assignment where each university is matched with its median set of students. If N is even, using the packing decomposition it is straightforward to get a similar result for markets with an even number of players: Just set $\alpha = 0.5 + \epsilon$ or $\alpha = 0.5 - \epsilon$, and each agent will be assigned to one of two median sets of partners. In the example in Figure 3, this corresponds to choosing all items crossed by the line $\alpha = 0.5$, i.e., occupying the halfway mark of every bin. The matching obtained: $u_1 \leftarrow \{1, 5, 8, 10\}$, $u_2 \leftarrow \{7\}$; $u_3 \leftarrow \{2, 4, 6\}$; $u_4 \leftarrow \{9, 11\}$; $u_5 \leftarrow \{3\}$. This corresponds to the solution M_4, which is stable. M_4 is thus the median stable solution.

This discussion can be summarized into the following result.

THEOREM 4.3. *There exists a stable solution to the many-to-one stable matching problem with the following properties*:

• *Assign each student to the median university, among all universities assigned to in the stable solutions*;

• *Assign each university its median set of students, among all stable admission solutions, and where the set of students assigned are ordered based on the ranking of the worst assigned student in each solution.*

PROOF. Let $\mu_1 >_u \mu_2 >_u \cdots >_u \mu_K$ be the distinct sets of students that are assigned to university u in some stable matching. Note that without loss of generality, we may assume that $|\mu_i| = q_u$ for all i. Let k_i be the number of times μ_i appears in all the stable solution. Hence $\sum_{i=1}^{K} k_i = N$. Suppose N is odd. The median solution is μ_{k^*} where

$$k^* = \operatorname{argmin}\left\{j: \sum_{i=1}^{j} k_i \geq \frac{N+1}{2}, \sum_{i=1}^{j-1} k_i \leq \frac{N-1}{2}\right\}.$$

As in the proof to Theorem 4.1, we examine the packing procedure by focusing on the phases where bins associated with u are being packed. We proved in Theorem 4.1 that in phase i, items in $\mu_i \setminus \mu_{i-1}$ are being packed into the bins. In the beginning of phase k^*, the maximum available space in the bins is greater than 0.5, but the addition of the items in $\mu_{k^*} \setminus \mu_{k^*-1}$ decreases the maximum available space on all bins to less than 0.5. Hence, the set of students assigned to u, under $I_{0.5}(u)$, are exactly the students corresponding to the items in $\mu_{k^*} \setminus \mu_{k^*-1}$, and $\mu_{k^*} \cap \mu_{k^*-1}$. Hence, the q_u students in $I_{0.5}(u)$ are exactly the students in μ_{k^*}. □

The existence of a matching mechanism that assigns all participants their median choice is indeed surprising. One of the limitations of this mechanism, though, is that all the stable matching solutions must be enumerated before a median solution can be constructed. This can theoretically be obtained by enumerating all the corner points of the polytope given by the LP characterization of the stable admission polytope or by combinatorial methods. Finding an efficient method to compute the median solution remains a challenging open problem.

4.2. Admissions with couples. Finally, we consider the stable admissions problem with couples. This is an extension of the stable admission problem to incorporate externalities, where the preferences of the students might depend on the matching produced—in particular, where the stable admission problem includes pairs of couples, whose preferences (ranking of the universities) also depend on where their spouses are assigned under the matching.

Consider, for instance, the example discussed in §3.2. Suppose further that a_4 and a_{10} are a "couple." Although both prefer university u_3 to the rest of the hospitals, their preferences could change if one of them is not assigned to u_3. In particular, if university u_3 is very far from u_1, whereas u_4 and u_5 are nearby, then a_4 and a_{10}, who prefer to stay together, might rank the assignment $\{(u_5, a_4), (u_4, a_{10})\}$ above the assignment $\{(u_3, a_4), (u_1, a_{10})\}$, although individually they preferred the assigned universities in the latter assignment.

To handle the externalities introduced by the preferences of the couples, one common approach is to allow the couples to submit joint rankings of the pairs of universities they to which would like to be assigned (cf. Roth and Sotomayor [32] and the references therein). We can define an analogous notion of stability in this problem. Unfortunately, many of the appealing results for the stable marriage problem do not carry over to the case where couples express preferences over pairs of positions. For example, (Aldershof and Carducci [3]):

(i) There are instances of the couples problem that do not have a stable matching;

(ii) Even if an instance of the couples problem has a stable matching, it may not have a university-optimal or student-optimal stable matching;

(iii) In an instance of the couples problem in which the preference lists are not complete, there may be stable matchings that leave different numbers of positions unfilled.

Klaus and Klijn [20] recently shows that stable solutions exist for the couple problem whenever the couples have "weakly responsive" preferences.[6] However, the definition of "weakly responsive" preferences essentially ensures that the couple preferences can be captured using suitably defined individual preferences, thus reducing the problem to the classical stable admissions problem without couples. This guarantees the existence of stable matching. Unfortunately, these assumptions on the preference profiles, which guarantee existence of a stable solution, may violate the most important consideration—that couples wish to be assigned to universities in close geographical proximity. Klaus and Klijn [20] further show that any slight deviation from the weakly responsiveness assumption easily results in instances where the stable admission problem will have no feasible solution. Interestingly, their example basically allows a couple to reverse their ranking, taking into account the geographical considerations. This suggests that any preference profiles submitted by couples that take into account the geographical considerations are likely to lead to instances where stable matching does not exist.

Cantala [11] also considers a class of realistic preferences in which (i) the "universities" are partitioned into regions; (ii) each couple first decides on a ranking of the regions (on which each couple agrees) and then ranks the universities within each region; their individual rankings of the universities within a given region may be different; and (iii) their preferences are otherwise independent, and each of them cares only about the university they to which are assigned. Even under such severe restrictions on the class of preferences, Cantala [11] shows that a stable matching might not exist.

The literature so far seems to indicate that the stable admissions problem with couples might not have feasible solutions if realistic preferences of the couples (based on geographical proximity of the joint assignment) are to be modeled. In the rest of this section, we propose a different approach to handle this problem: Instead of allowing the couples to submit joint ranking of the universities we accept only individual ranking of the universities, and treat the problem as a classical stable admissions problem. We model the preferences of the couples indirectly as side constraints. We look for stable solutions in the college admissions problem where the couples are matched to colleges in the same geographical region.

We use a simple example to motivate our approach. Consider the example discussed in §3.2, where the individual preferences for the universities are as shown in Table 1. Suppose further that (a_4, a_{10}), (a_2, a_8) are couples. They would like to be assigned to universities in close proximity, so that they can stay together. For the couple (a_4, a_{10}), as far as possible, they would like both to be assigned to universities in the cluster $\{u_3\}$, or to the cluster $\{u_1, u_2, u_4, u_5\}$. Similarly, the couple (a_2, a_8) would like both to be assigned to universities in the cluster $\{u_1, u_3\}$, or to the cluster $\{u_2, u_4, u_5\}$.

To incorporate these additional considerations, we need to answer the questions: "Is there a stable matching that meets these additional requirements? If so, how do we construct the solution?"

The stable matching M_1 fails to address the concerns of the couples, because a_4 is assigned to u_3, but a_{10} is assigned to u_1. The matching M_7 satisfies the additional requirement of couple (a_4, a_{10}) (both are matched to universities in the same cluster $\{u_1, u_2, u_4, u_5\}$), but fails to address the needs of couple (a_2, a_8). There is only one matching in this problem that meets the additional requirement from both couples: M_5.

In the rest of this section, we show that under a natural, restricted class of preferences, one can efficiently determine whether or not a stable matching exists simply by solving a related linear program. In particular, as long as the concerns of the couples can be modeled by geographical concerns similar to (a)–(c) defined below, our model can be used to determine whether a feasible solution exists, and if so, construct a feasible solution.

Specifically, we assume that each couple partitions the universities into several groups based on their geographical locations. For example, the couple might choose to group together all of the universities located in the same city. Note that the grouping of the universities can be different for different couples. In our previous example, for instance, the couple (a_4, a_{10}) has chosen to group the universities into clusters $\{u_3\}$, $\{u_1, u_2, u_4, u_5\}$, whereas the couple (a_2, a_8) has chosen to group the universities into clusters $\{u_1, u_3\}$ and $\{u_2, u_4, u_5\}$. Each husband and wife submit separate rankings of the universities subject to the following restriction:

(a) The ranking is first done by ranking the group and then ranking the universities in the group;

(b) If a person prefers Group 1 to Group 2, then all universities deemed acceptable[7] in Group 1 will be ranked above any university deemed acceptable in Group 2; and

(c) Each couple must rank the groups in the same way, but are allowed to submit different rankings of the universities within a group.

For each couple c, let $\mathcal{K}_c = K_{c,1} \cup K_{c,2} \cup \cdots \cup K_{c,D}$ denote the groupings of the (acceptable) universities, sorted according to the ranking of the cities, i.e., (acceptable) universities in group $K_{c,i}$ are generally preferred

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[6] cf. B. Klaus and F. Klijn (2005): Corrigendum: Stable matchings and preferences of couples, for a minor correction.

[7] Certain universities in the group may not be acceptable to the student.

by the couple c to universities in group $K_{c,j}$, if $i < j$. The essential geographic preferences of the couples are captured by allowing the husband and wife to be assigned to different universities as long as these universities are within the same group (i.e., in the same region). This will be modeled as a side constraint in our model.

For ease of exposition, we assume that each couple has partitioned the universities into D groupings. Furthermore, we will assume that the students accept all universities (i.e., complete ranking) to which they have been assigned. This condition can easily be relaxed by adding additional constraints of the type $x_{u,a} = 0$ if student a does not accept university u.

We wish to find a stable matching in which each couple is assigned to universities within the same group. We show that a variation of the linear inequality description for the stable admission polytope remains valid for this version of the stable matching problem with couples; in particular, a stable solution, if it exists, can be found using linear programming.

We claim that the following inequalities give the polytope of stable admission with couples (P_{SAC}):

$$\sum_{u:\,(u,a)\in\Gamma} x_{u,a} \le 1, \quad \forall\, a \in A, \tag{15}$$

$$\sum_{a:\,(u,a)\in\Gamma} x_{u,a} \le q_u, \quad \forall\, u \in U, \tag{16}$$

$$x(C) = \sum_{(u,a)\in C} x_{u,a} \ge q_u, \quad \forall\, C \in C_u,\ u \in U. \tag{17}$$

$$x_{u,a} \ge 0 \quad \forall\, a,\ (u,a) \in \Gamma, \tag{18}$$

$$\begin{cases} \displaystyle\sum_{j\in K_{c,l}} x_{p,j} + \sum_{j\notin K_{c,l}} x_{q,j} \le 1 \\[2mm] \displaystyle\sum_{j\in K_{c,l}} x_{q,j} + \sum_{j\notin K_{c,l}} x_{p,j} \le 1, \end{cases} \quad \text{for all couples } (p,q) \text{ and group } K_{c,l}. \tag{19}$$

Here the inequalities (15), (16), (17), (18), and (19) are called the student, university, comb, nonnegativity, and the couple constraints, respectively. The first four classes of constraints are basically the valid constraints in the classical many-to-one stable admissions polytope. As long as both students in the couple pair are matched to some universities, then the couple constraints ensure that they are matched to universities in the same geographical locations.

The fractional solution of (P_{SAC}) still satisfies the decomposition property of §3.1, because it is a subset of the stable admission polytope. Note that if a student is unmatched in the stable admission problem (without the added side constraints), then he/she remains unmatched in all the stable matchings. This property ensures that either $\sum_u x_{u,a} = 0$ or $\sum_u x_{u,a} = 1$ in (P_{SAC}).

Consider the randomized rounding method described in Theorem 3.3: Generate a number $\alpha \in (0,1]$ uniformly at random; for each university u, assign the q_u students in $I_u^{\alpha}(x)$ to university u to get an assignment μ_α. Note that our rounding method will automatically produce a stable matching for the original problem. We need to ensure further that it satisfies the additional couple constraints. This is clear whenever $\sum_u x_{u,p} = 0$ or $\sum_u x_{u,q} = 0$ for couple (p,q). On the other hand, if $\sum_u x_{u,p} = 1$ and $\sum_u x_{u,q} = 1$, then the couple constraints reduce to

$$\sum_{j\in K_{c,l}} x_{p,j} \le \sum_{j\in K_{c,l}} x_{q,j},$$

and

$$\sum_{j\in K_{c,l}} x_{q,j} \le \sum_{j\in K_{c,l}} x_{p,j},$$

i.e.,

$$\sum_{j\in K_{c,l}} x_{p,j} = \sum_{j\in K_{c,l}} x_{q,j} \quad \text{for all } K_{c,l}.$$

Because the individual preferences of p and q are obtained by an identical ranking of the groupings first, the above property ensures that in our randomized rounding method, whenever p is assigned to a university in $K_{c,l}$, q will be assigned to one in $K_{c,l}$, too.

We thus have the following result.

THEOREM 4.4. *The convex hull of the stable admission solutions to the couples problem satisfying conditions* (a), (b), *and* (c) *is given by* (P_{SAC}).

When $(P_{SAC}) = \varnothing$ (when the LP is infeasible), the LP method will return an inconsistent set of inequalities automatically, producing a certificate to the fact that there does not exist any stable solution assigning all couples to the same geographical regions.

The above model for the couples problem inherit all the nice properties for the classical stable admission problem, whenever a feasible solution can be found. In particular, the lattice properties of the feasible stable admission solutions are preserved under this model. Using the same argument in our derivation of the median solution concept, and the fact that $(P_{SAC}) \subset (P_{SA})$, we also have the next result.

THEOREM 4.5.   *If* $(P_{SAC}) \neq \varnothing$, *then a median solution exists for the stable admission problem with couples.*

5. Concluding remarks.   In this paper, we study the stable admissions problem using a simple geometric property of the fractional solutions in the stable admissions polytope. We exhibit a simple decomposition property that allows us to express fractional solutions in the stable admissions polytope as convex combinations of integral solutions. The results provide new insight into the stable admissions problem and generalize many structural properties of the classical stable marriage problem.

The stable admissions problem can be viewed as a special case of the stable marriage problem with indifference. However, because the problem of determining whether a given pair is stable is NP-complete for a given instance of stable marriage with ties (Manlove et al. [26]), this appears to be substantially more difficult. It is interesting to find out whether the geometric structure of the fractional solutions carries over to this situation. Another special case of the stable matching problem with indifference is the many-to-many stable matching problem with strict preferences. We believe that the properties of the fractional solutions has an analogous interpretation in this general setting. However, describing the stable matching polytope explicitly in this case requires a nontrivial extension to the comb inequalities. Another interesting open issue is to consider more general preferences for the universities for which the existence of a stable matching is assured; the class of *substitutable preferences* is a natural candidate. We leave these issues for further research.

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